Problem for Introduction to Computation. Battle of Waterloo.

Consider yourself Napoleon at the battle of Waterloo of 18 June 1815. You are fighting against the troops of General Wellington. This exercise shows that history might have turned a different direction had Napoleon had the computation power of modern age.
Behind the lines of Napoleons troops stayed a battery of canons aimed beyond his own vanguard troops at the army of Wellington, see the Figure on the next page. The canons were fired, but it takes a long time to find the exact angle of firing. With some expertise, the trial and error method of finding the target can be reduced in time.

On the other hand, we can also calculate the angle.
Consider the figure on the next page. The vanguard troops have told us that the distance between the canons and Wellington's troops is exactly 1700 m. Unfortunately we have no control over the initial speed of the canon ball, which is fixed at 500 km/h and can only change the angle. The question is: where will it land? We don't want to overshoot and we certainly don't want to hit our own troops. To calculate we will make use of the following Newtonian laws:

$$\text{(I)}\ \frac{dv_y}{dt} = -g \qquad \text{(II)}\ \frac{dy}{dt} = v_y \qquad \text{(III)}\ \frac{dx}{dt} = v_x$$

v_y is vertical speed [m/s]
v_x is horizontal speed [m/s]
t is time [s]
g is gravitational constant = 9.81 [m/s^2]

For this situation without friction the solution can also be calculated analytically; the resulting trajectory is a parabola. If for instance the canon and the targets are in the same level ($y_{\text{end}} = y_0 = 0$) we can reason as follows: At the top of the parabola the vertical velocity v_y is zero. We can easily determine at what time we reach this maximum height. From I and II: $v_y = v_{y0} - g\,t$, thus $t_{\text{top}} = v_{y0}/g$. When the ball is back on the ground, after 2 t_{top} it has traveled a horizontal distance of $2\,t_{\text{top}}\,v_{x0} = 2\,v_{x0}\,v_{y0}/g$.

- Write a program that calculates how far the cannonball reaches with as input the angle (0^o is horizontal). Take as time integration step 1 ms. Use units according to SI (m, s, etc.). For 30^o the ball reaches _________m.
- Add a loop to the program that scans the angle (steps of 1^o). The furthest we reach is with an angle of _______^{o}. We hit Wellington's army for an angle of _______^{o}.

Often it is advantageous to be located in the mountains. Now we will see why.

- Change the program such that the canon is located 300 m above the enemy. What is the maximum reach now? _______ m for an angle of ________°.

Now assume that the air causes friction. This adds the following contributions

$$\text{(IV)}\ \frac{dv_y}{dt} = -fv_y \qquad \text{(V)}\ \frac{dv_x}{dt} = -fv_x$$

- Put the canon back on the ground level. Assume a friction of f = 0.01 s^{-1}. Change the program to include friction. With an angle of 45° we reach _________ m. The angle to hit the enemy army is now __________°.
- With a wind in the back (20 m/s) and with the canon set with the angle found above, we will miss the enemy by _______ m.

Optional: In reality the friction is not linear with the speed, as described above, but cubic:

$$\text{(VI)}\ \frac{d|v|}{dt} = -f_3|v|^3$$

Implement the above friction with f_3 = 0.001 s/m^2. How far does the ball get for an angle of 45°? Answer: _______ m.