Complex numbers

MIEET 1º ano

Peter Stallinga UAlg 2011

Complex numbers

What are complex numbers and how can they help us in (Electronic) Engineering?

Definition

$z = x + i\,y$ ex. $2 - i$

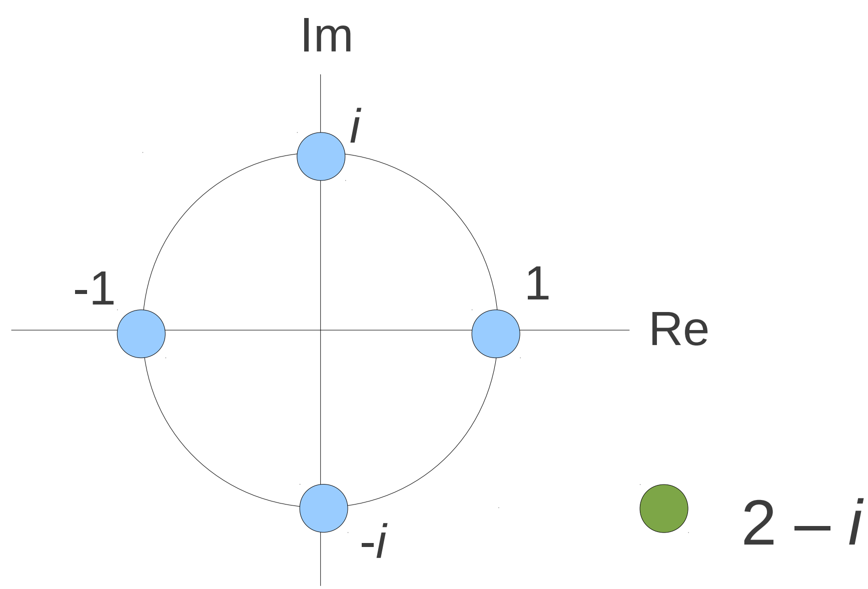
$i^2 = -1$

$x = \mathrm{Re}(z)$

$i = \sqrt{-1}$

$y = \mathrm{Im}(z)$

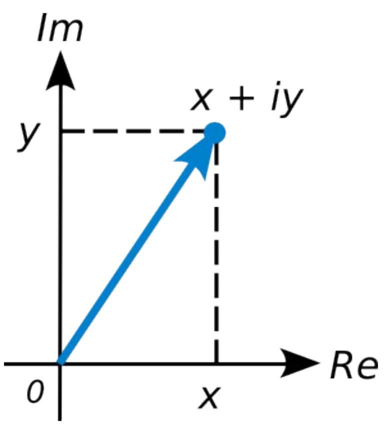
Visualization:

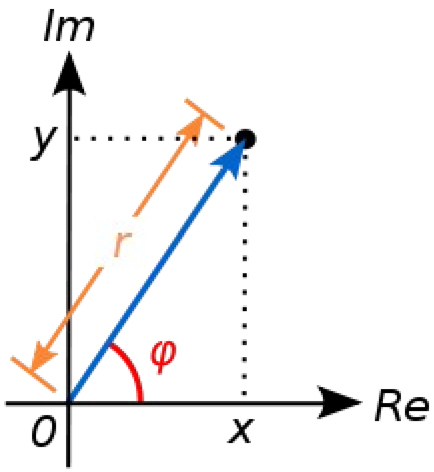


Cartesian vs. Radial

$z = x + i\,y$

$z = r\,e^{i\varphi}$



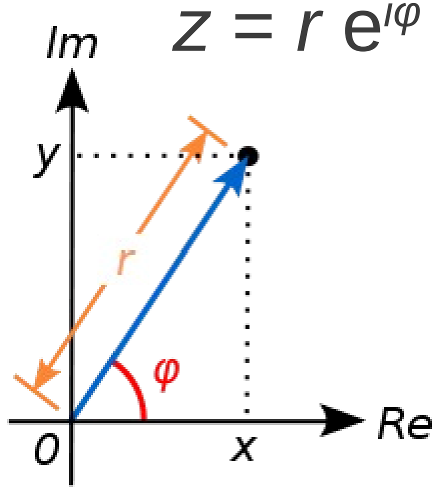


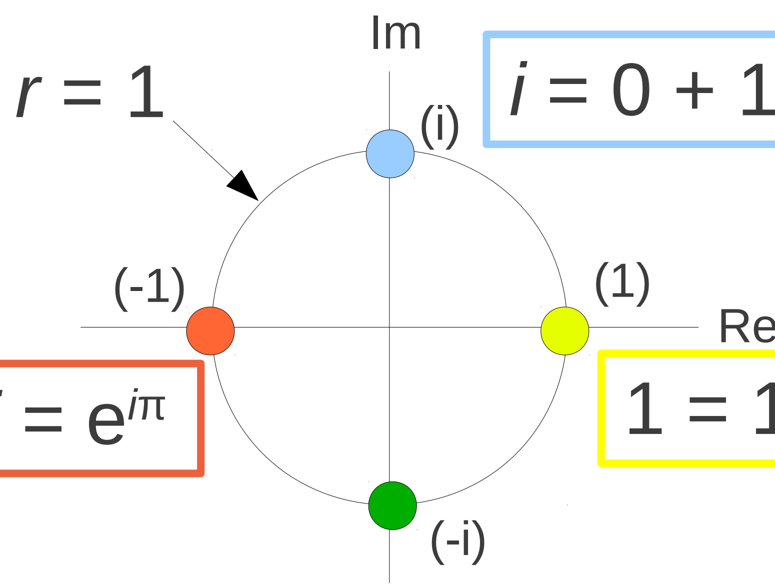
$x = r\cos(\varphi)$
$y = r\sin(\varphi)$

$r = |z| = \sqrt{(x^2+y^2)}$
$\varphi = \tan^{-1}(y/x)$

Examples

$z = r\, e^{i\varphi}$





$i = 0 + 1\,i = e^{i\pi/2}$

$-1 = -1 + 0\,i = e^{i\pi}$

$1 = 1 + 0\,i = e^{0}$

$-i = 0 - 1\,i = e^{-i\pi/2}$

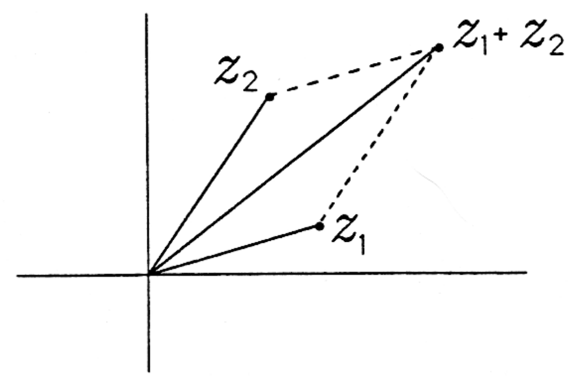
Adding and multiplying

$$z_1 = x_1 + i\,y_1 \qquad z_1 = r_1\, e^{i\varphi 1}$$

$$z_2 = x_2 + i\,y_2 \qquad z_2 = r_2\, e^{i\varphi 2}$$

$$z = z_1 + z_2 = (x_1 + x_2) + i\,(y_1 + y_2)$$

Add



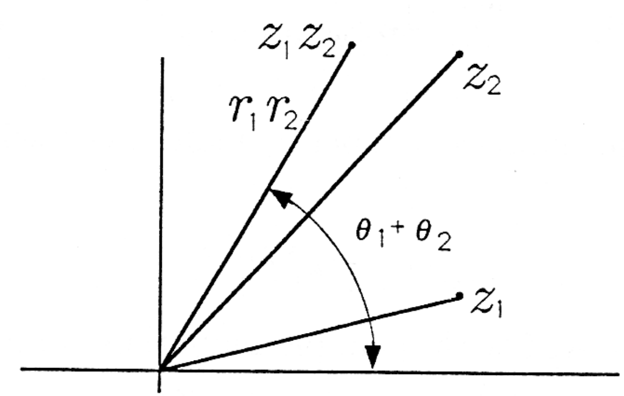
$$z = z_1 \times z_2 = (x_1 + i\,y_1) \times (x_2 + i\,y_2)$$

Multiply

$$= (x_1x_2 - y_1y_2) + i\,(x_1y_2 + x_2y_1)$$

i^2

$$= r_1\, r_2\, e^{i\varphi 1}\, e^{i\varphi 2} = r_1\, r_2\, e^{i(\varphi 1+\varphi 2)}$$



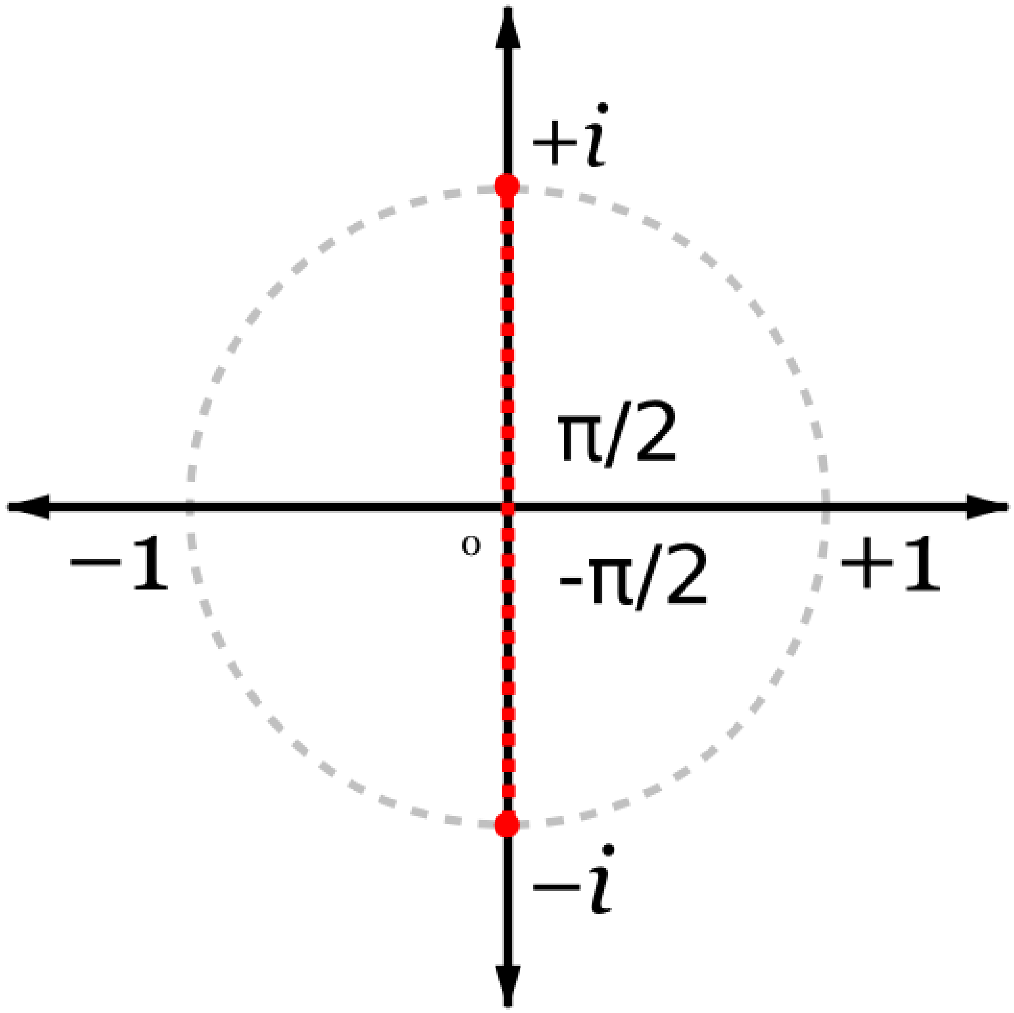
Example 1

What is $\sqrt{-1}$?

$z^2 = -1$

$= -1 + 0\, i = 1 \times e^{i\pi}$

$r^2 = 1,\ 2\varphi = \pi \pmod{2\pi}$



$\varphi = \pi/2,\ r = \sqrt[2]{1} = 1$: $z = \cos(\pi/2) + i \sin(\pi/2) = i$

$\varphi = -\pi/2,\ r = \sqrt[2]{1} = 1$: $z = \cos(-\pi/2) + i \sin(-\pi/2) = -i$

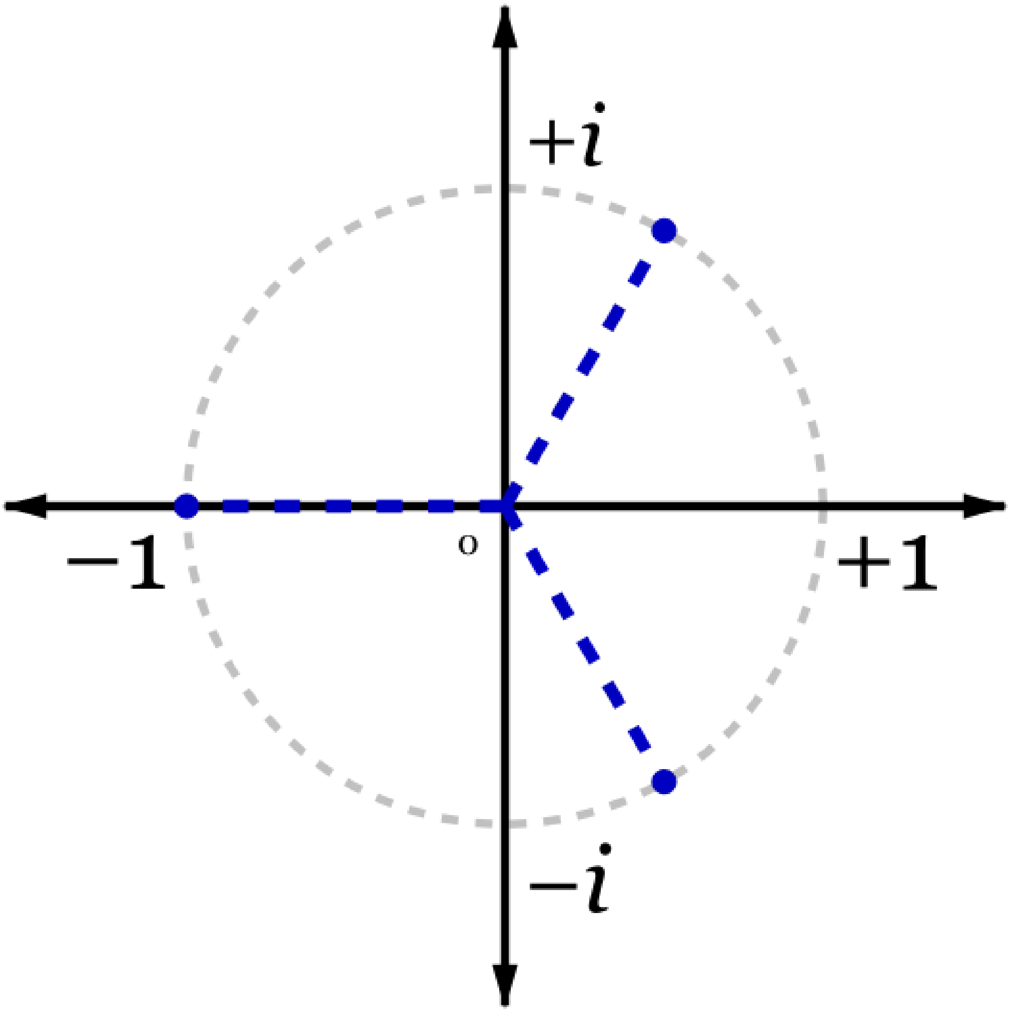
Example 2

What is $\sqrt[3]{-1}$?

$z^3 = -1$

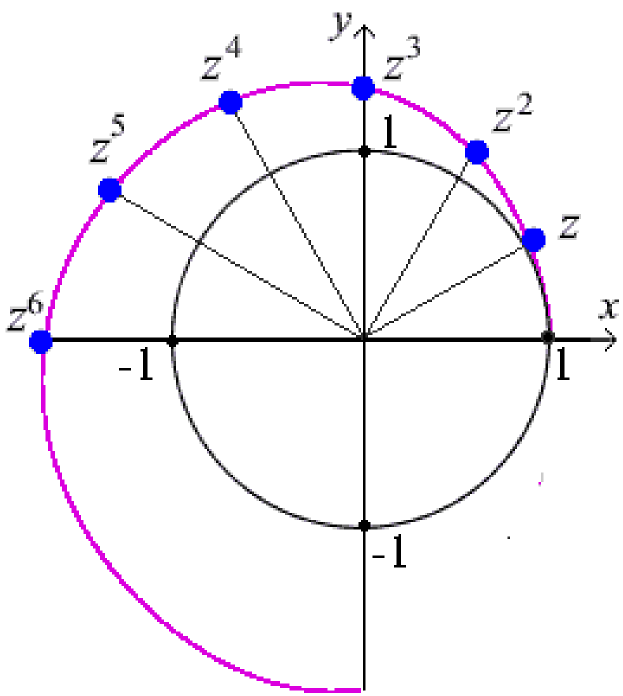
$= -1 + 0\,i = 1 \times e^{i\pi}$

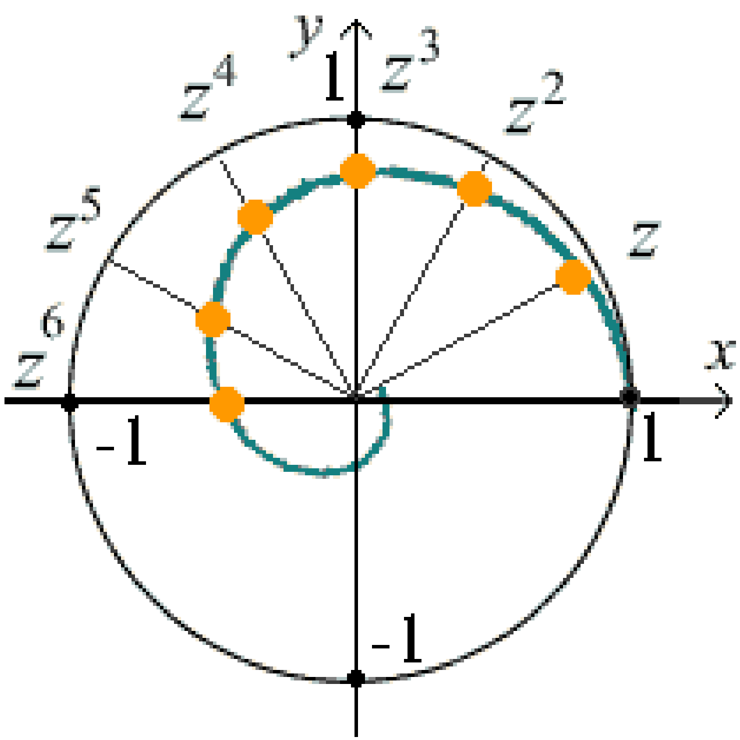
$r^3 = 1$, $3\varphi = \pi \pmod{2\pi}$



$\varphi = \pi/3$, $r = \sqrt[3]{1} = 1$: $z = \cos(\pi/3) + i \sin(\pi/3) = ½ + i\, ½\sqrt{3}$
$\varphi = -\pi/3$, $r = \sqrt[3]{1} = 1$: $z = \cos(-\pi/3) + i \sin(-\pi/3) = ½ - i\, ½\sqrt{3}$
$\varphi = \pi$, $r = \sqrt[3]{1} = 1$: $z = -1$

Examples





$|z| > 1$ $|z| < 1$

Revision: Current of a capacitor

If $Q = C \times V$, then: changes of voltage cause changes of stored charge:

$$\Delta Q = C \times \Delta V$$

How fast we do it matters

$$\Delta Q/\Delta t = C \times \Delta V/\Delta t$$

In the mathematical limit:

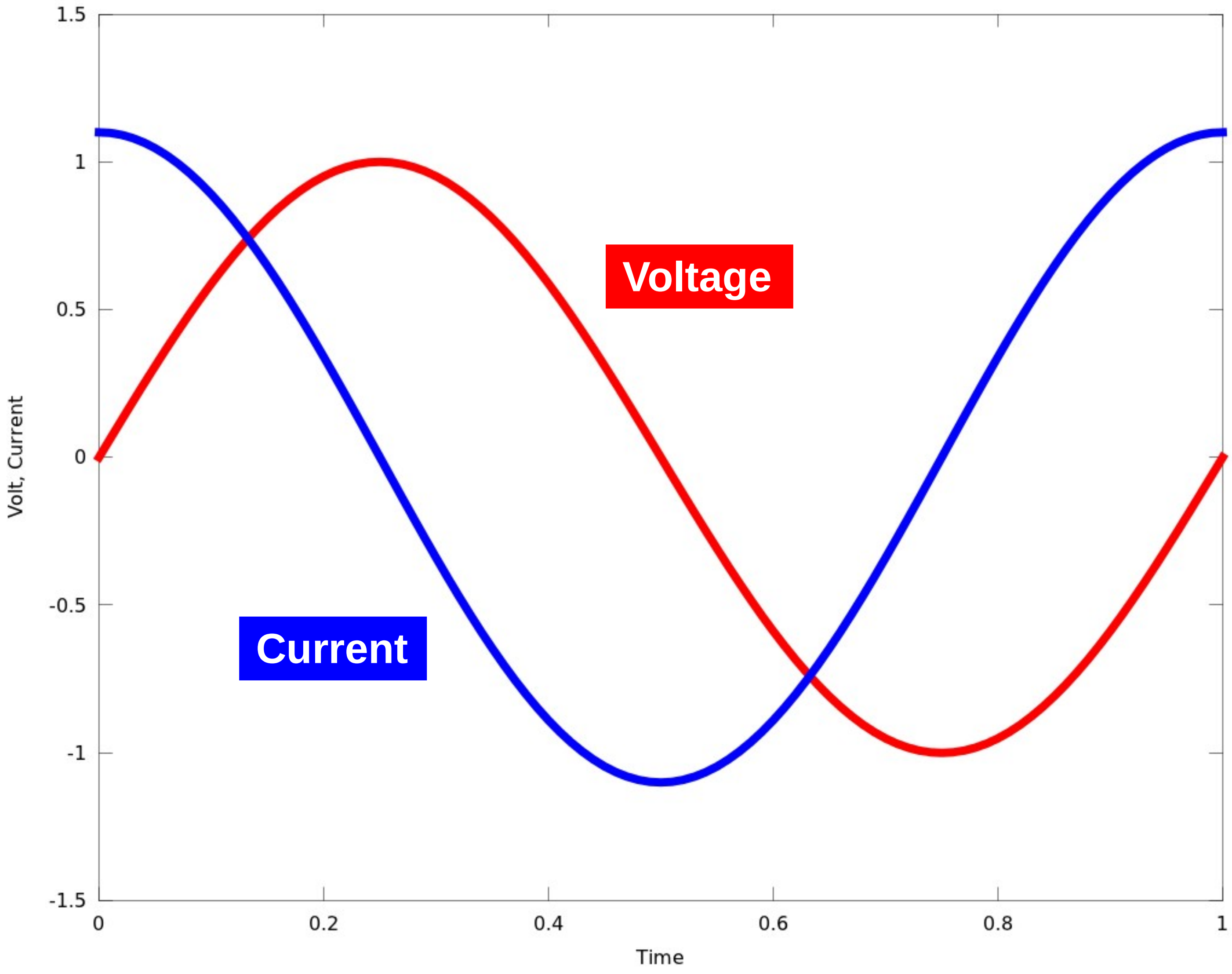
$$\mathrm{d}Q/\mathrm{d}t = C \times \mathrm{d}V/\mathrm{d}t$$

But, the left side is the **definition of current**

$$I = C \times \mathrm{d}V/\mathrm{d}t$$

Current in a capacitor is proportional to the speed of changes of the applied voltage

Revision: Capacitance example

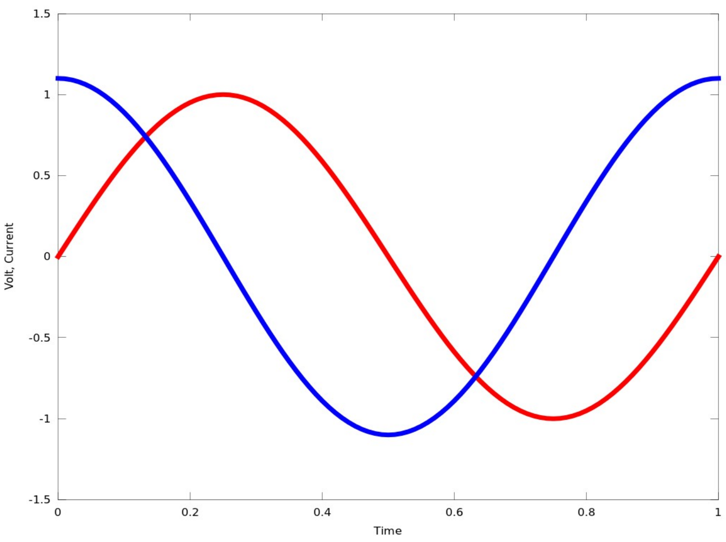


Current of a capacitor

$I(t) = C \times dV(t)/dt$

$V(t) = \cos(\omega t)$

$I(t) = -\omega C \sin(\omega t)$



Current of a capacitor

$I(t) = C \times dV(t)/dt$

$V(t) = \cos(\omega t)$

$I(t) = -\omega C \sin(\omega t)$

$I(t) = C \times dV(t)/dt$

$V(t) = \cos(\omega t)$
$= \mathrm{Re}[\cos(\omega t) + i \sin(\omega t)]$
$= \mathrm{Re}[e^{i\omega t}]$

$I(t) = C \times d/dt\, [e^{i\omega t}]$?
$= C \times i\omega\, e^{i\omega t}$
$= C \times i\omega \times [\cos(\omega t) + i \sin(\omega t)]$
$= C \times \omega \times [-\sin(\omega t) + i \cos(\omega t)]$

$V(t) = \mathrm{Re}[V_{complex}(t)]$
$I(t) = \mathrm{Re}[C\, dV_{complex}(t)/dt]$

The real world

$V(t) = \mathrm{Re}[V_{complex}(t)]$

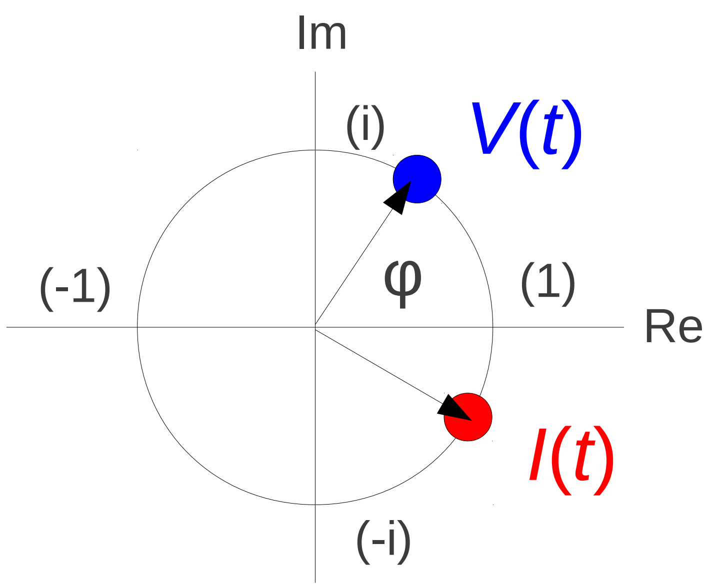
$I(t) = \mathrm{Re}[C\,\mathrm{d}V_{complex}(t)/\mathrm{d}t]$

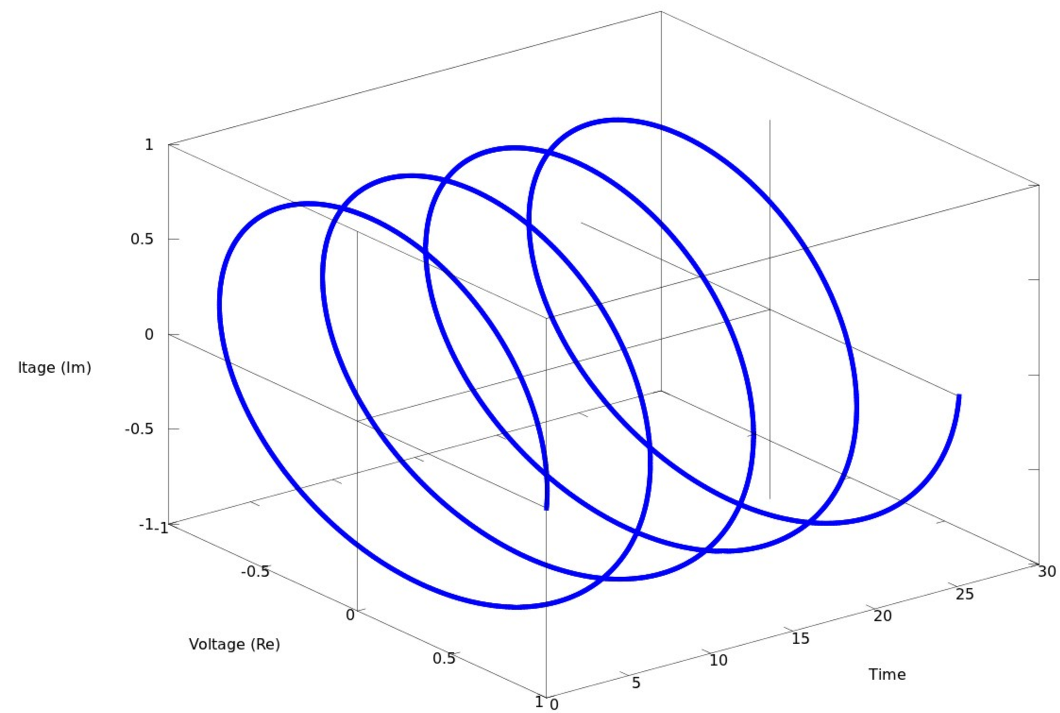
We can use complex numbers, but don't forget that measurable quantities such as voltage and current are **always real**!

Interesting things are never complex. Complex numbers only help us in the calculations

Imaginary numbers are there to help us do the bookkeeping

Voltage and current in a sable dance





$$\varphi(t) = \omega t$$

$$V(t) = \cos(\omega t)$$

$$I(t) = -\omega C \sin(\omega t)$$

'Resistance' of a capacitor

Wait, it gets even better. What about Ohm's Law?

$$R = V/I$$

$V(t) = e^{i\omega t}$

$I(t) = C \times i\omega\, e^{i\omega t}$

$$R_C = V/I = \frac{1}{i\omega C}$$

This complex 'resistance' is called impedance

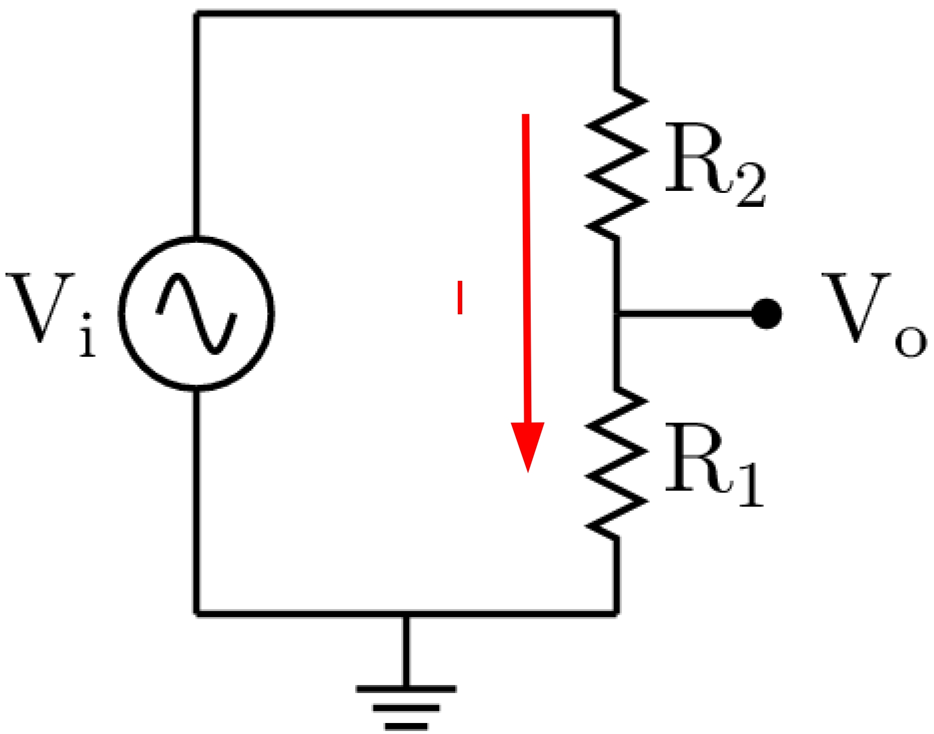
Frequency response of a capacitor

$$R_C = V/I = \frac{1}{i\omega C}$$

Low frequencies:
For $\omega = 2\pi f = 0$: 'resistance' is infinite: no current

High frequencies:
For $\omega = 2\pi f = \infty$: 'resistance' is zero: short circuit

Voltage divider



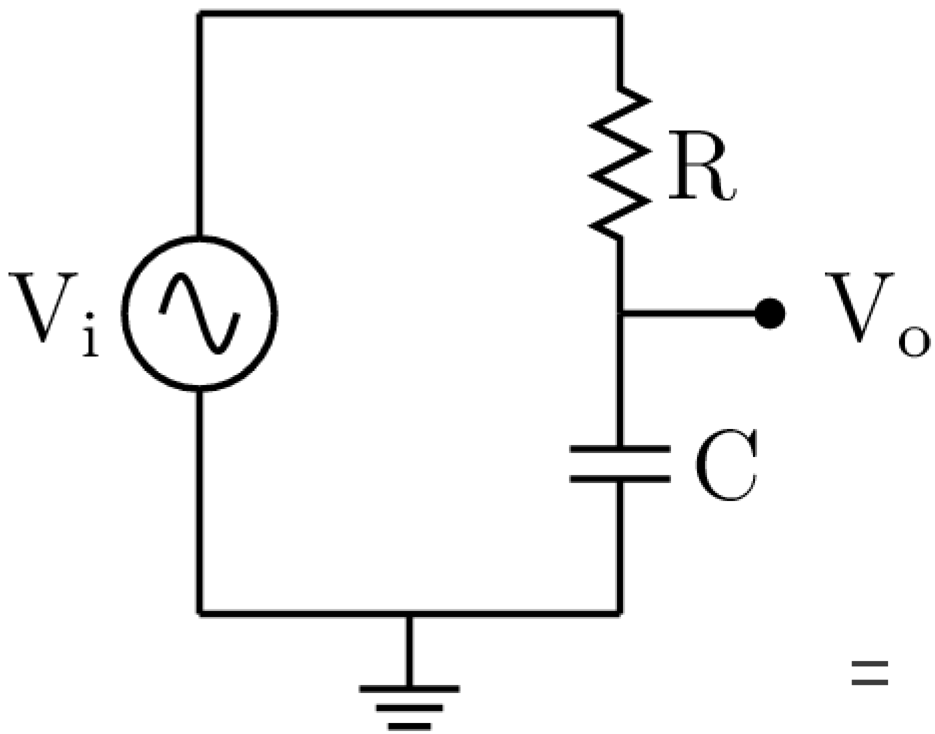
What is the voltage halfway?

Resistances in series: $R = R_1 + R_2$

$I = V_i / R = V_i / (R_1 + R_2)$

$$V_o = 0 + I \times R_1 = V_i \frac{R_1}{R_1 + R_2}$$

A simple filter



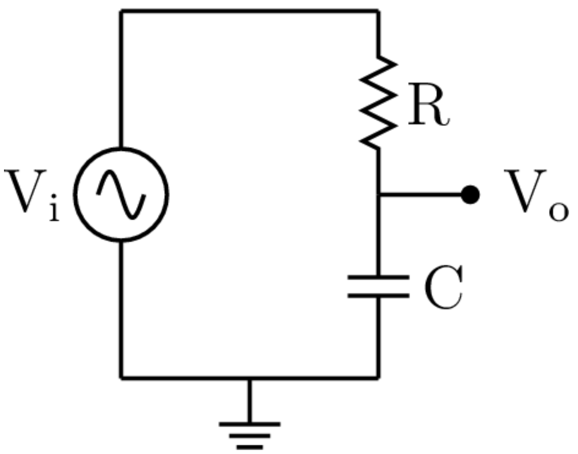
$$V_o/V_i = \frac{R_1}{R_1 + R_2}$$

$$= \frac{R_C}{R + R_C}$$

$$= \frac{1/i\omega C}{R + 1/i\omega C} = \frac{1}{i\omega RC + 1}$$

A simple filter

$$V_o/V_i = \frac{1}{i\omega RC + 1}$$



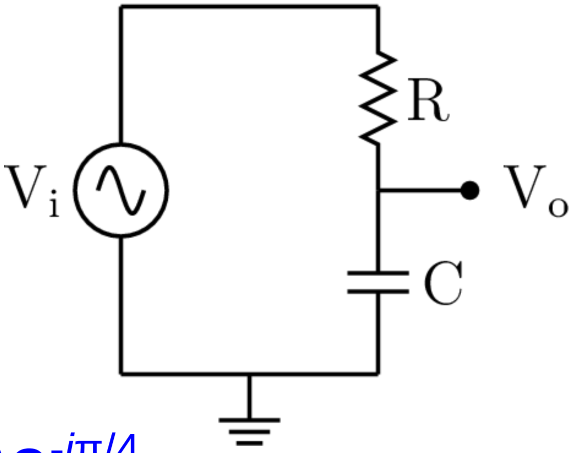
$\omega = 0$: $V_o/V_i = 1$

$\omega = \infty$: $V_o/V_i = 0$

$\omega = 1/RC$: $V_o/V_i = \frac{1}{i + 1}$

A simple filter

$$V_i = \cos(\omega t) = \mathrm{Re}[\cos(\omega t) + i\sin(\omega t)] = \mathrm{Re}[e^{i\omega t}]$$

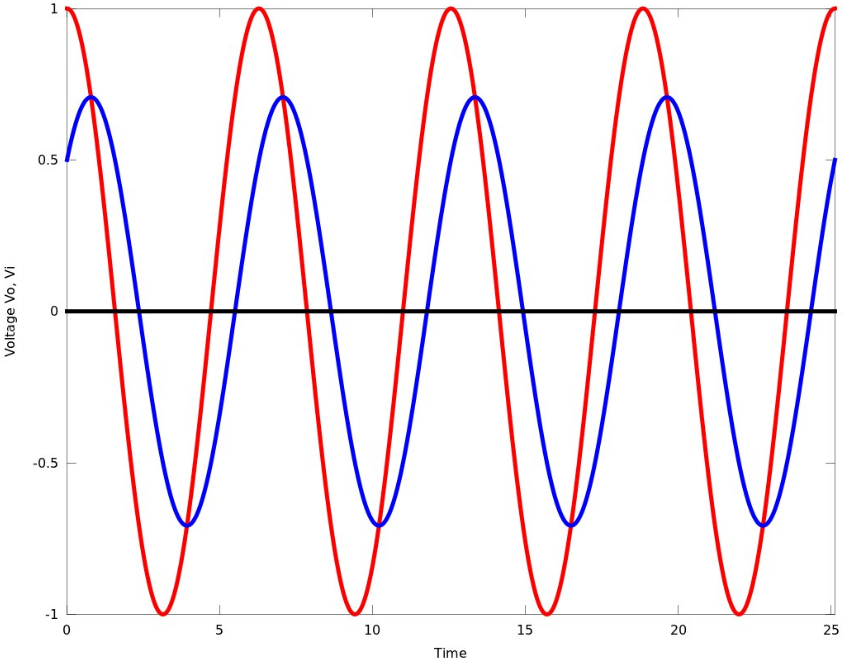


$\omega = 0$: $V_o/V_i = 1 \quad \rightarrow \quad V_o = \cos(\omega t)$

$\omega = \infty$: $V_o/V_i = 0 \quad \rightarrow \quad V_o = 0$

$\omega = 1/RC$: $V_o/V_i = \dfrac{1}{i+1} = ½[1 - i\,] = (1/\sqrt{2})e^{-i\pi/4}$

$$V_o = \mathrm{Re}[\,V_o/V_i \times V_i\,]$$
$$= \mathrm{Re}[(1/\sqrt{2})e^{-i\pi/4}\, e^{i\omega t}]$$
$$= (1/\sqrt{2})\cos(\omega t - \pi/4)$$



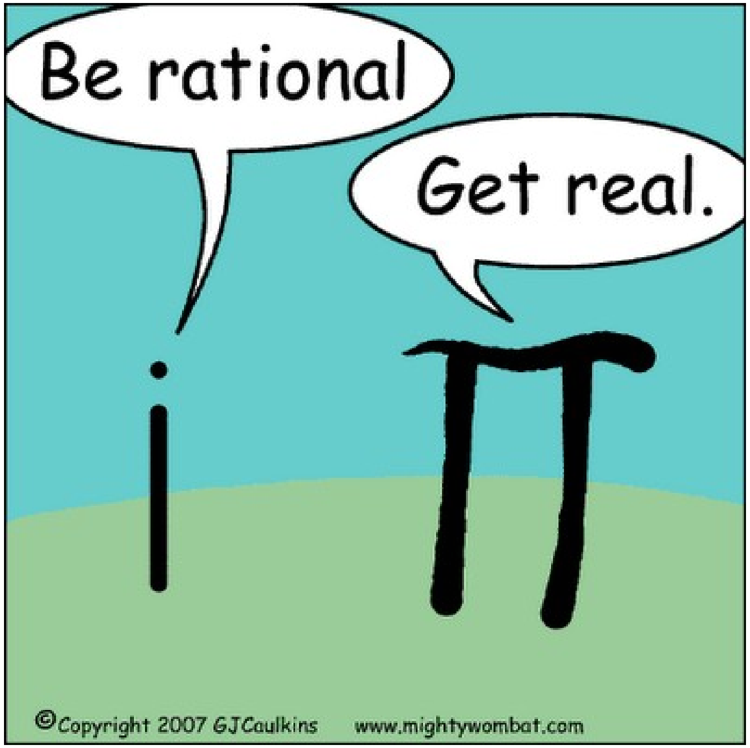
With frequency, amplitude is dropping and a delay is introduced

Summary

• Complex numbers are a useful tool to make complex things simple

• The **real world** is the **real** part of the calculations

• The **imaginary** part is only a tool to help us and **has nothing to do with reality**



1 = -1

$$
\begin{aligned}
-1 &= -1 \\
-1/1 &= -1/1 \\
-1/1 &= 1/-1 \\
\sqrt{-1/1} &= \sqrt{1/-1} \\
i/1 &= 1/i \\
i &= 1/i \\
i^2 &= 1 \\
-1 &= 1
\end{aligned}
$$